Suggested Solutions to Midterm Test for MATH4220

March 9, 2017

1. (**20 points**)

   (a) (**10 points**) Find all the solutions to

   $$u_x - 2u_y + 2u = 1$$

   (b) (**10 points**) Solve the problem

   $$\begin{cases} y\partial_x u + 3x^2 y \partial_y u = 0 \\ u(x=0,y) = y^2 \end{cases}$$

   In which region of the xy-plane is the solution uniquely determined?

Solution:

(a) **Method 1:Coordinate Method**:
Change variables to

$$x' = x - 2y,\ y' = -2x - y$$

Hence $u_x - 2u_y + 2u = 5u_{x'} + 2u = 1$. Thus the solution is $u(x', y') = f(y')e^{-\frac{2}{5}x'} + \frac{1}{2}$, with f an arbitrary function of one variable. Therefore, the general solutions are

$$u(x,y) = \frac{1}{2} + f(-2x - y)e^{-\frac{2}{5}(x-2y)}$$

where f is an arbitrary function.
Method 2: Geometric Method
The corresponding characteristic curves are

$$\frac{dx}{1} = \frac{dy}{-2}$$

that is, $y = -2x + C$ where C is an arbitrary constant.Then

$$\frac{d}{dx}u(x, -2x + C) = u_x(x, -2x + C) - 2u(x, -2x + C) = -2u(x, -2x + C) + 1$$

Hence $u(x, -2x + C) = f(C)e^{-2x} + \frac{1}{2}$, where f is an arbitrary function. Therefore,

$$u(x,y) = \frac{1}{2} + f(2x + y)e^{-2x}$$

where f is an arbitrary function.

(b) The characteristic curves are

$$\frac{dy}{3x^2 y} = \frac{dx}{y}$$

that is, $y = x^3 + C$ where C is an arbitrary constant. Then

$$\frac{d}{dx}u(x, x^3 + C) = u_x + 3x^2 u_y = 0$$

Hence $u(x, x^3 + C) = f(C)$ where f is an arbitrary function. Thus

$$u(x, y) = f(y - x^3)$$

Besides, the auxiliary condition gives that $y^2 = u(x = 0, y) = f(y)$. Hence, the solution is

$$u(x, y) = (y - x^3)^2$$

Note that when $y = 0$ the equation vanishes, thus the characteristic curves break down when $y = 0$, therefore the solution is uniquely determined on $\{(x, y) : y > 0, y > x^3\} \cup \{(x, y) : y < 0, y < x^3\} \cup \{(0, 0)\}$. (Remark: if the solution is **continuous**, then u is uniquely determined on the whole plane by the continuity of u).

2. (**20 points**)

   (a) (**4 points**) What is the type of the equation

$$\partial_t^2 u + \partial_{xt}^2 u - 2\partial_x^2 u = 0 \ ?$$

   (b) (**16 points**) Solve the Cauchy problem

$$\begin{cases} \partial_t^2 u + \partial_{xt}^2 u - 2\partial_x^2 u = 2, & -\infty < x < +\infty, \quad -\infty < t < +\infty \\ u(x, t = 0) = x^2, \quad \partial_t u(x, t = 0) = 0, & -\infty < x < +\infty \end{cases}$$

Solution:

(a) Since $a_{11} = 1, a_{12} = \frac{1}{2}, a_{22} = -2$, then $a_{12}^2 - a_{11}a_{22} = \frac{9}{4} > 0$, hence it is hyperbolic.

(b) Let

$$t = t', \ x = \frac{1}{2}t' + \frac{3}{2}x'$$

and $v(x', t') = u(x, t)$, then v satisfies

$$\begin{cases} \partial_{t'}^2 v - \partial_{x'}^2 v = 2, \\ v(x', t' = 0) = (\frac{3}{2}x')^2, \quad \partial_{t'} v(x', t' = 0) = \frac{3}{2}x', \end{cases}$$

Thus d'Alembert formula gives that

$$\begin{aligned} v(x', t') &= \frac{1}{2}\Big\{\frac{9}{4}(x' + t')^2 + \frac{9}{4}(x' - t')^2)\Big\} + \frac{1}{2}\int_{x'-t'}^{x'+t'} \frac{3}{2} y dy + \frac{1}{2}\int_0^{t'} \int_{x'-(t'-s)}^{x'+(t'-s)} 2 dy ds \\ &= \frac{13}{4}t'^2 + \frac{9}{4}x'^2 + \frac{3}{2}x't' \end{aligned}$$

Then

$$u(x, t) = v(x', t') = v(\frac{2}{3}x - \frac{1}{3}t, t) = 3t^2 + x^2.$$

3. (**20 points**)

   (a) (**5 points**) State the definition of a well-posed PDE problem.

   (b) (**5 points**) Is the following problem well-posed? Why?

$$\begin{cases} \dfrac{d^2 u}{dx^2} + \dfrac{du}{dx} = 1, \qquad 0 < x < 1 \\ u'(0) = 1, u'(1) = 0 \end{cases}$$

(c) (**10 points**) State and prove the uniqueness and continuous dependence of solutions to the problem

$$\begin{cases} \partial_t u = \partial_x^2 u, \quad 0 < x < 1, \quad 0 < t < T, \quad T > 0 \\ \partial_x u(0,t) = 0, \partial_x u(1,t) = 0, \qquad t > 0 \\ u(x, t = 0) = \phi(x), \quad 0 \le x \le 1 \end{cases}$$

Solution:

(a) A PDE problem is said to be well-posed if the following three properties are satisfied:
Existence: There exists at least one solution $u(x,t)$ satisfying all these conditions.
Uniqueness: There is at most one solution.
Stability: The unique solution $u(x,t)$ depends in a stable manner on the data of the problem. This means that if the data are changed a little, the corresponding solution changes only a little.

(b) The problem is **not** well-posed, since the solution doesn't exist.
Indeed, the general solution to the ODE $\frac{d^2u}{dx^2} + \frac{du}{dx} = 1$ is $u(x) = C_1 + C_2 e^{-x} + x$. Then $u'(x) = -C_2 e^{-x} + 1$, and boundary condition $u'(0) = 1$ implies that $C_2 = 0$, however $u'(1) = 1 \neq 0$. Thus the solution cannot exist.

(c) **Uniqueness:** Let u_1 and u_2 be any two solutions to the problem, then $u_1 = u_2$.
Continuous dependence on initial data: If u_1 and u_2 are solutions to the problem with initial condition $\phi_1(x)$ and $\phi_2(x)$ respectively, then

$$\sup_{0 \le t \le T} \int_0^1 |u_1 - u_2|^2 dx \le \int_0^1 |\phi_1 - \phi_2|^2 dx.$$

Proof:
(Uniqueness): Let $v = u_1 - u_2$, then v satisfies

$$\begin{cases} \partial_t v = \partial_x^2 v, \quad 0 < x < 1, \quad 0 < t < T, \ T > 0 \\ \partial_x v(0,t) = 0, \partial_x v(1,t) = 0, \qquad t > 0 \\ v(x, t = 0) = 0, \quad 0 \le x \le 1 \end{cases}$$

Multiplying the both sides of $\partial_t v = \partial_x^2 v$ by v and taking intergral from 0 to 1 with respect to x, then we have

$$\int_0^1 \partial_t v v dx = \int_0^1 \partial_x^2 v v dx$$

Then

$$L.H.S = \frac{d}{dt} \int_0^1 \frac{1}{2} v^2 dx$$

$$R.H.S = \partial_x v v \Big|_0^1 - \int_0^1 (\partial_x v)^2 dx = - \int_0^1 (\partial_x v)^2 dx \le 0$$

Then, we have for $t > 0$

$$0 \le \int_0^1 \frac{1}{2} v^2(x,t) dx \le \int_0^1 \frac{1}{2} v^2(x,0) dx = 0$$

By the continuity of v, we have $v(x,t) \equiv 0, \ 0 < x < 1, 0 < t < T$. Thus we have shown that $u_1(x,t) \equiv u_2(x,t)$ for $0 < x < 1, \ \ 0 < t < T$.
(Continuous dependence on initial data:) Let $v = u_1 - u_2$, then v satisfies

$$\begin{cases} \partial_t v = \partial_x^2 v, \quad 0 < x < 1, \quad 0 < t < T, \ T > 0 \\ \partial_x v(0,t) = 0, \partial_x v(1,t) = 0, \qquad t > 0 \\ v(x, t = 0) = \phi_1(x) - \phi_2(x) =: \tilde{\phi}(x), \quad 0 \le x \le 1 \end{cases}$$

Multiplying the both sides of $\partial_t v = \partial_x^2 v$ by v and taking intergral from 0 to 1 with respect to x, then we have

$$\int_0^1 \partial_t v v dx = \int_0^1 \partial_x^2 v v dx$$

Then

$$L.H.S = \frac{d}{dt}\int_0^1 \frac{1}{2}v^2 dx$$

$$R.H.S = \partial_x v v\Big|_0^1 - \int_0^1 (\partial_x v)^2 dx = -\int_0^1 (\partial_x v)^2 dx \le 0$$

Then, we have

$$\sup_{0\le t\le T}\int_0^1 \frac{1}{2}v^2(x,t)dx \le \int_0^1 \frac{1}{2}v^2(x,0)dx = \int_0^1 \frac{1}{2}\tilde{\phi}^2(x)dx$$

which completes the proof of the continuous dependence on initial data.

4. (**20 points**)

(a) (**15 points**) Derive the solution formula for the following initial-boundary value problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < +\infty, \quad t > 0 \\ u(x, t=0) = \phi(x) & 0 < x < +\infty \\ \partial_x u(x=0,t) = 0, & t > 0 \end{cases}$$

by the method of reflection (with all the details of the derivation).

(b) (**5 points**) Let $\phi(x) = \cos x, 0 < x < +\infty$. Find the maximum value of $u(x,t)$.

Solution:

(a) Use the reflection method, and first consider the following Cauchy Problem:

$$\begin{cases} \partial_t v = \partial_x^2 v, & 0 < x < +\infty, \quad t > 0 \\ v(x,t=0) = \phi_{even}(x) & 0 < x < +\infty \end{cases}$$

where $\phi_{even}(x)$ is even extension of ϕ which is given by

$$\phi_{even}(x) = \begin{cases} \phi(x), & \text{if } x > 0 \\ \phi(-x), & \text{if } x < 0 \end{cases}$$

Then the unique solution is given by:

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{even}(y)dy$$

And since $\phi_{even}(x)$ is even, so is $v(x,t)$ for $t > 0$,which implies

$$\partial_x v(x=0,t) = 0, t > 0$$

Set $u(x,t) = v(x,t), x > 0$,then $u(x,t)$ is the unique solution of Neumann Problem on the half-line. More presicely, $x > 0, t > 0$

$$\begin{aligned} u(x,t) &= \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_{-\infty}^0 S(x-y,t)\phi(-y)dy \\ &= \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_0^{\infty} S(x+y,t)\phi(y)dy \\ &= \frac{1}{\sqrt{4k\pi t}}\int_0^{\infty}[e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}]\phi(y)dy. \end{aligned}$$

Here, $k = 1$.

(b) By (a), the solution is given by ($k = 1$)

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty [e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}] \cos y dy$$

then

$$|u(x,t)| \leq \frac{1}{\sqrt{4k\pi t}} \int_0^\infty [e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}] |\cos y| dy \leq \frac{1}{\sqrt{4k\pi t}} \int_0^\infty [e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}] dy = 1$$

That is, $|u(x,t)| \leq 1$. Note that $\max_{0<x<\infty} u(x,0) = \max_{0<x<\infty} \cos x = 1$, which implies that 1 can be attained by u. Hence

$$\max_{0<x<\infty, t>0} u(x,t) = 1.$$

5. (**20 points**)

(a) (**10 points**) Prove the following generalized maximum principle:
If $\partial_t u - k\partial_x^2 u \leq 0$ on $R \triangleq [0,l] \times [0,T]$ with a positive constant k, then

$$\max_R u(x,t) = \max_{\partial R} u(x,t)$$

here $\partial R = \{(x,t) \in R |$ either $t = 0$, or $x = 0$, or $x = l\}$.

(b) (**10 points**) Show if $v(x,t)$ solves the following problem

$$\begin{cases} \partial_t v = k\partial_x^2 v + f(x,t), & 0 < x < l, 0 < t < T \\ v(x,0) = 0, & 0 < x < l \\ v(0,t) = 0, v(l,t) = 0, & 0 \leq t \leq T \end{cases}$$

with a continuous function f on $R \triangleq [0,l] \times [0,T]$. Then

$$v(x,t) \leq t \max_R |f(x,t)|$$

(Hint, consider $u(x,t) = v(x,t) - t \max_R |f(x,t)|$ and apply the result in (a).)

Solution:

(a) Let $v(x,t) = u(x,t) + \epsilon x^2$, then v satisfies

$$\partial_t v - k\partial_x^2 v = \partial_t u - k\partial_x^2 u - 2k\epsilon < 0$$

First, **claim** that v attains its maximum on the parabolic boundary R. Let $\max_R v(x,t) = M = v(x_0, t_0)$. Suppose on the contrary, then either

i. $0 < x_0 < l, 0 < t_0 < T$.
In this case, $v_t(x_0,t_0) = v_x(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \leq 0$. Thus $\partial_t v - k\partial_x^2 v\big|_{(x_0,t_0)} \geq 0$, which is impossible.

ii. $0 < x_0 < l, t_0 = T$.
In this case, $v_t(x_0,t_0) \geq 0, v_x(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \leq 0$. Thus $\partial_t v - k\partial_x^2 v\big|_{(x_0,t_0)} \geq 0$, which is impossible.

Hence

$$\max_R v(x,t) = \max_{\partial R} v(x,t).$$

Then for any $(x,t) \in R$,

$$u(x,t) \leq u(x,t) + \epsilon x^2 \leq \max_{\partial R} v(x,t) \leq \max_{\partial R} u(x,t) + \epsilon l^2$$

Letting $\epsilon \to 0$ gives $u(x,t) \leq \max_{\partial R} u(x,t)$ for any $(x,t) \in R$. Hence $\max_R u(x,t) = \max_{\partial R} u(x,t)$.

(b) Let $u(x,t) = v(x,t) - t \max_R |f(x,t)|$, then u satisfies

$$\begin{cases} \partial_t u - k\partial_x^2 u = -\max\limits_R |f(x,t)| + f(x,t) \leq 0 \\ u(x,0) = 0, \\ u(0,t) = u(l,t) = -t \max\limits_R |f(x,t)| \leq 0 \end{cases}$$

Hence the result in (a) implies that for any $(x,t) \in R$,

$$u(x,t) \leq \max_{\partial R} u(x,t) = 0$$

that is, $v(x,t) \leq t \max_R |f(x,t)|$.